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Topology and its Applications 153 (2006) 3437–3449

**Topology
and its
Applications**

www.elsevier.com/locate/topol

Inverse limit spaces arising from problems in economics

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Received 13 October 2004; accepted 8 March 2006

Abstract

In this paper we use tools from topology and dynamical systems to analyze the structure of solutions to implicitly defined equations that arise in economic theory, specifically in the study of so-called “backward dynamics”. For this purpose we use inverse limit spaces and shift homeomorphisms to describe solutions which are typical in that they are likely to be observed in future time. These predicted solutions corresponds to attractors in an inverse limit space under the shift homeomorphism(s).

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MSC: 37B45; 37E05; 54F15; 54H20

Keywords: Economics; Attractor; Invariant set; Inverse limits; Unimodal; Continuum; Indecomposable

1. Introduction

In applications of dynamical system theory, the first step is usually the definition of a discrete- or continuous-time model with a view to understanding the evolution of certain variables of interest and forecasting their future.

In discrete-time, the canonical form of such a model is a difference equation like

$$x_{t+1} = F(x_t; \mu) \quad (1)$$

where x is a vector of state variables; $t \in \mathbb{Z}^+$ is a time index; F is a map defined on a certain state space, e.g. a subset of \mathbb{R}^n ; and μ denotes a vector of structural parameters. Eq. (1) is interpreted as the law of motion of the system, derived from “first principles” or from observation, and the purpose of the investigation is to study the orbits generated by iterations of F as well as the changes in the orbit structure occurring when one or more parameters are varied. The forward limit set of the map F , describing the asymptotic properties of its dynamics, is usually interpreted as the hypothetical long-run time evolution of the system. When the limit set is complicated, however, and includes many (even uncountably many) periodic orbits as well as aperiodic and chaotic sets, we face the problem of selecting among them.

Consider for example the much-studied family of logistic maps on the interval $F_\mu(x) = \mu x(1 - x)$. As is well known, in the interval $\mu_F < \mu \leq 4$ (where $\mu_F \approx 3.58 \dots$ corresponds to the so-called Feigenbaum limit), there exist

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values of μ for which the iterations of F_μ generate very complicated dynamics. For each of these “chaotic values”, e.g. $\mu = 4$, there exist very many compact subsets of $[0, 1]$ invariant with respect to F_μ , including a set of periodic points dense in $[0, 1]$. In such a case, why do we say that the map F_μ is chaotic, rather than periodic? The simple answer is: because we know that for Lebesgue-most initial conditions $x_0 \in [0, 1]$ the orbit generated by F_μ and starting from x_0 converges to a compact, F_μ -invariant set Λ , on which F_μ is topologically transitive and generates very complicated dynamics characterized by sensitive dependence on initial conditions which is the distinctive mark of chaos. On the other hand, the set $A \subset [0, 1]$ of initial conditions such that the orbits starting from $x_0 \in A$, are attracted to a periodic orbit Γ is negligible in the sense that A has Lebesgue measure zero. Notice that this excludes that A may be open nonempty and this in turn implies that arbitrarily near a point whose orbit converges to Γ , there is some other point whose orbit diverges from it. Sometimes this situation is described by saying that Λ is an attractor, but Γ is not. This is also the reason why, when we perform numerical simulations of chaotic maps, we expect the screen of our computer to show complicated, not periodic orbits.¹

The strategy of distinguishing typical from exceptional *forward* orbits of a map according to the measure of basins of attraction, as subsets of the variable space, works only if there is a one-to-one correspondence between initial conditions and forward orbits of the map. There are certain problems in applications, however, for which that condition fails and different criteria must be followed. For example, there is a vast class of problems in the theory of intertemporal economic equilibrium that give rise to models whose typical form is

$$H(x_t, x_{t+1}) = 0 \quad (2)$$

where the difference equation linking present and future values of the state variable x is defined only implicitly. Whether the function H can be inverted with respect to x_{t+1} or not depends on the fundamental functional relationships (typically, utility or production functions) implicit in (2), and there is no a priori reasons for assuming that Eq. (2) can always be re-written in the form of (1). An interesting case, commonly found in economics, occurs when H , although not invertible with respect to x_{t+1} , can be inverted with respect to x_t , leading to the difference equation:

$$x_t = F_\mu(x_{t+1}) \quad (3)$$

which is formally identical to (1), except that now the dynamics generated by F_μ is backwards in time. Mathematically, Eq. (3) can be studied in exactly the same manner as (1), but the interpretation of results is fundamentally different. After all, in real life agents worry about the future not the past and therefore they are concerned with forward not backward dynamics. Hence the problem arises whether the investigation of the backward dynamics generated by a dynamical model like (3) could be used to understand the general properties of the forward dynamics implicitly defined by it.

To give the flavor of the situation that we have in mind, we shall provide a concise description of the simplest example of backward-in-time dynamics drawn from the family of “overlapping generations models” (OLG) on which there exists an enormous economic literature.²

Consider an economy with a constant population living two periods of time, and, at each period, being divided into two equally numerous classes of persons, labelled respectively “young” and “old”. Because individuals in each class are assumed to be identical, we shall describe the situation in terms of “the young (old) representative agent”. There is no production but fixed amounts of the unique, perishable consumption good are distributed at the beginning of each period to young and old.

For each generation, the young representative agent maximizes utility of consumption over his/her two-period life, subject to an intertemporal budget constraint, i.e. for each generation, the total value of consumption must be no greater than the total value of the endowments received. We also assume that the market for the consumption good is always in equilibrium, i.e. in each period the demand for the consumption good from young and old is equal to the total supply, namely the total endowments, and that agents’ expectations are always fulfilled (perfect foresight).

Let $c_t \geq 0$ be the young agent’s consumption at time t , and let $g_t \geq 0$ be the old agent’s consumption at time t . Let $w_0 \geq 0$ and $w_1 \geq 0$ be the young and old agent’s endowment respectively. Let $\rho_t > 0$ be the interest factor at

¹ Of course, this is a naïve expectation and, strictly speaking, it would be exactly fulfilled only in an ideal situation in which computers were infinitely precise and initial points could actually be chosen at random.

² Two articles directly relevant to the present discussion are [5] and [10]. A recent general introduction to the subject can be found, for example, in [1].

time t , i.e. the exchange rate between present and future consumption. Define the utility functions by $U(c_t, g_{t+1}) = u_1(c_t) + u_2(g_{t+1})$ with $u'_i(\cdot) > 0$ and $u''_i(\cdot) \leq 0$ for $i = 1, 2$ (the same for all agents).

A mathematical formulation of the problem then is to maximize the function:

$$u_1(c_t) + u_2(g_{t+1}) \quad (4)$$

such that $g_{t+1} \leq w_1 + \rho_t(w_0 - c_t)$ and $c_t, g_{t+1} \geq 0$. The market-clearing condition for all time t is:

$$c_t + g_t = w_0 + w_1. \quad (5)$$

Economists are interested in studying the properties of *infinite* sequences of c_t (or, equivalently, of g_t), satisfying the optimality conditions (4) and the market-clearing condition (5). One must add a non-negativity condition $c_t, g_t \geq 0$, for all t , because negative consumption makes no sense economically.

From the first order conditions of the constrained maximization and condition (5), we deduce that the young agent's optimal choice must satisfy the equation:

$$H(g_{t+1}, c_t) = \mathcal{U}(g_{t+1}) + \mathcal{V}(c_t) = 0 \quad (6)$$

where $\mathcal{U}(g) = u'_2(g)(w_1 - g)$ and $\mathcal{V}(c) = u'_1(c)(w_0 - c)$.

Whether or not from (6) we can derive a difference equation moving forward in time depends on whether the function \mathcal{U} is invertible. Consider the following specific example:

$$u_1(c) = ac - (b/2)c^2; \quad u_2(g) = g \quad (7)$$

where a, b are positive constants. In this case, $\mathcal{U}(g) = w_1 - g$ is of course invertible. For simplicity's sake and without loss of generality, we put $w_0 = 0$ and $a = b = \mu$. Then we can write:

$$w_1 - g_{t+1} = c_{t+1} = F_\mu(c_t) = \mu c_t(1 - c_t) \quad (8)$$

a much-studied non-invertible map. Starting from an arbitrary initial condition $c_0 \in [0, 1]$, Eq. (8) determines sequences of young agents' consumption forward in time. Using the equilibrium condition (5), it will determine old agents' consumption as well.³

Suppose now that we interchange the utility functions (7) and we put $w_1 = 0, w_0 > 0$.⁴ In this case, instead of (8) we can write an equation

$$w_0 - c_t = g_t = F_\mu(g_{t+1}) \quad (9)$$

where the map F_μ is as before but it now defines sequences of old agents' consumption (and by implication, young agents' consumption) moving backward in time.

This is not a mere technical point, though. Economically, it means that to each value of the young agent's present saving (endowment–consumption) there may correspond two or more values of his/her expected future consumption that justify it—or *there may be none*.⁵ This problem will occur with great generality whenever the function \mathcal{U} is non-invertible. Nor is it limited to this version of OLG but it may also occur in two-dimensional models of OLG with production, i.e., models in which consumption is produced by means of current labour and capital stock invested one period ago.⁶ In order to have a visual insight of the nature of the problem when Eq. (9) holds, let us consider Fig. 1, where we represent the curve of the function F_μ in the plane (g_{t+1}, g_t) (and, for simplicity's sake, we assume that $g_{\max} = w_0$).

Start at time $t = 0$, and suppose that our maximizing young agent considers the possibility of consuming an amount c_0 and thereby saving an amount $w_0 - c_0 = g_0$. The counter-image of $g_0 \in [0, g_{\max} = w_0]$, $F_\mu^{-1}(g_0)$, consists of two

³ In this case, at each t , the young agent's consumption, c_t , must be equal the old agent's saving $w_1 - g_t$. Whereas w_0 is unimportant here, w_1 must be chosen in such a way that whenever $c_t \geq 0, g_t \geq 0$ too, i.e. $w_1 \geq c_{\max} = \mu/4$. Notice also that, for this choice of the utility functions, the model makes economic sense only for $c \in [0, 1]$.

⁴ In this case, at each t , the old agent's consumption must be equal to the young agent's saving, $w_0 - c_t$. Therefore, the restriction discussed in footnote 3 now becomes $w_0 \geq g_{\max}$.

⁵ A given amount of the young agent's current consumption, c_t and the associated current saving $w_0 - c_t = g_t$ is said to be “justified” by the same agent's future consumption g_{t+1} when the pair (g_t, g_{t+1}) corresponds to a point of the curve defined by the function F_μ , i.e., it maximizes (4) subject to the budget constraint and the market clearing condition (5). The curve defined by F_μ is sometimes called an “offer curve”.

⁶ See, for example, [18, pp. 221–239].

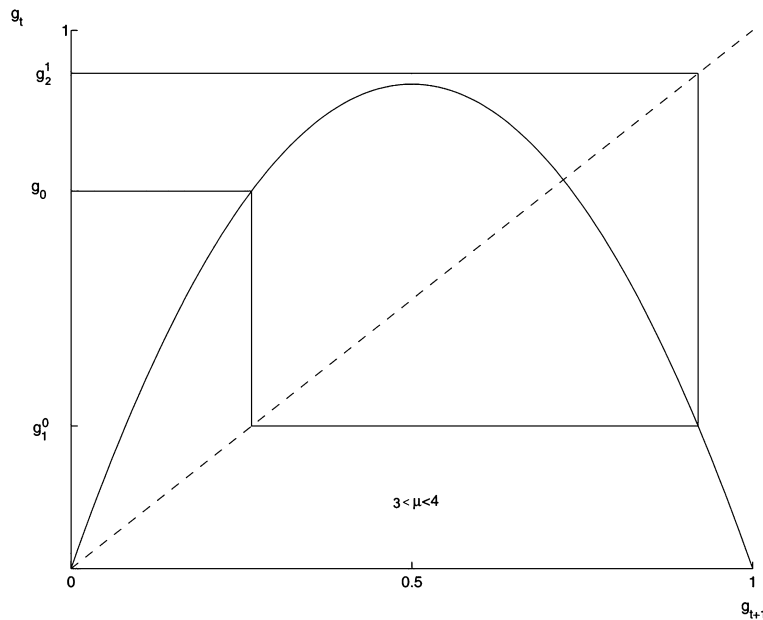


Fig. 1. Backward moving offer curve F_μ with $\mu < 4$.

points that we shall label, g_1^0 *low-level consumption* and g_1^1 *high-level consumption*, associated, respectively, with the rising and the falling branches of the graph of F_μ . In words, that means that there are two levels of future consumption at time 1 that would justify the young agent's consumption (and saving) decision at time 0. Analogously there are two levels of consumption at time 2 justifying a given level of consumption and saving at time 1, and so on and so forth. Suppose that, for whatever reasons, g_1^0 and g_1^1 are chosen. Inspection of Fig. 1 shows that the young agent at time $t = 2$ will never decide to consume the amount $c_2 = w_0 - g_2^1$ necessary to fulfill the old agent's expectation. Indeed, the counter-image of g_2^1 is empty, meaning that there is no amount of expected consumption at time $t = 3$ that, if realized, would justify the saving decision $w_0 - c_2 = g_2^1$.

This simple example indicates that, for the model at hand, not all infinite sequences of consumption are admissible, and prompts us to write the following definition:

Definition 1.1. An infinite sequence $\{g_t\}$ is said to be *forward (backward, forward and backward) admissible* if $g_t \geq 0$ and for each pair $(t, t + 1)$ $g_t = F_\mu(g_{t+1})$ for all $t \in \mathbb{Z}^+$ (respectively for all $-t \in \mathbb{N}$, for all $t \in \mathbb{Z}$). The admissibility of sequences $\{c_t\}$ can be defined similarly.

For example, in the case illustrated in Fig. 1, all the sequences $\{g_i^0\}_{i=0}^N$ (i.e., sequences along which the low-level alternative is selected at all times) are forward admissible for any $g_0 \in [0, \mu/4]$ and any $N > 0$. In fact, any forward admissible sequence starting at $g_0 < F_\mu^2(0.5)$ must contain only low-level terms. It is evident that these sequences tend asymptotically to a non-trade steady state $\bar{g}^{(1)} = \bar{c}^{(1)} - w_0 = 0$. Consider now the intersection between the curve of F_μ and the bisector, defining the non-trivial fixed point $\bar{g}^{(2)} = 1 - (1/\mu)$ of the map F_μ . For the assumed value of μ the slope of the curve at the intersection is greater than one in absolute value, i.e. the fixed point is unstable for the backward moving map F_μ , and therefore, if we choose g_0 sufficiently close to $\bar{g}^{(2)}$, a forward sequence $\{g_i^1\}_{i=0}^N$ is admissible for any $N > 0$, and the sequence converges to $\bar{g}^{(2)}$ as $N \rightarrow \infty$. Many more examples of forward admissible sequences for F_μ or similar non-invertible maps could be constructed.

In economic literature, the canonical strategy adopted to deal with the problem of backward dynamics is first to locate a non-trivial stationary solution (a fixed point of the relevant map), next to invoke the implicit function theorem and invert the map around the fixed point and finally to perform some local analysis of the system. Although this strategy has produced a number of useful results, it restricts the investigation severely, leaving out most interesting types of admissible forward orbits compatible with the dynamical system under investigation.

2. Inverse limit theory

In this paper we apply a well-understood tool from topology, the inverse limit space, to analyze all of the admissible forward orbits in the setting discussed in the previous section. We provide some of the background definitions and results from the theory of inverse limit spaces. For a more thorough treatment see [11] or [13]. See [16] for standard definitions from topology.

Definition 2.1. For each positive integer i let X_i be a topological space, (called a *factor space*), and let $f_i : X_{i+1} \rightarrow X_i$ be a continuous function (called a *bonding map*). The sequence $\{X_i, f_i\}_{i \in \mathbb{Z}^+}$ is called an *inverse sequence*, and the *inverse limit space* is defined by:

$$\varprojlim \{X_i, f_i\} = \left\{ (z_1, z_2, \dots) \in \prod_{i=1}^{\infty} X_i : f_i(z_{i+1}) = z_i \right\}.$$

We give the inverse limit space the topology inherited as a subspace of $\prod_{i=1}^{\infty} X_i$. If each factor space, X_i , is a metric space with metric, d_i , bounded by 1, then we have the following as a metric on $\varprojlim \{X_i, f_i\}$:

$$d[x, y] = \sum_{i=1}^{\infty} \frac{d_i[x_i, y_i]}{2^i}.$$

Definition 2.2. For each integer i we define the *projection map* $\pi_i : \varprojlim \{X_i, f_i\} \rightarrow X_i$ by $\pi_i[(z_1, z_2, \dots)] = z_i$.

Given two positive integers m and n with $n > m$ we use f_m^n to denote the map $f_m \circ f_{m+1} \circ \dots \circ f_{n-1} : X_n \rightarrow X_m$. It is easy to see that

$$\pi_m(z) = f_m^n \circ \pi_n(z), \quad (10)$$

for every $z \in \varprojlim \{X_i, f_i\}$.

Fix a positive integer N . Given an inverse sequence $\{X_i, f_i\}$ let $\{Y_i, g_i\}$ be another inverse sequence with $Y_i = X_{i+N}$ and $g_i = f_{i+N}$. We will denote $\varprojlim \{Y_i, g_i\}$ by simply $\varprojlim \{X_{i+N}, f_{i+N}\}$.

Definition 2.3. The natural homeomorphism from $\varprojlim \{X_i, f_i\}$ to $\varprojlim \{X_{i+N}, f_{i+N}\}$, is called the N th *shift map* $\sigma_N : \varprojlim \{X_i, f_i\} \rightarrow \varprojlim \{X_{i+N}, f_{i+N}\}$ and is defined by $\sigma_N[(z_1, z_2, \dots)] = (z_{N+1}, z_{N+2}, \dots)$.

We will denote σ_1 by σ . Notice that for $n > m$ we immediately have

$$\pi_m \circ \sigma_{n-m}(z) = \pi_n(z). \quad (11)$$

In much of the recent literature regarding inverse limit spaces each factor space X_i is simply some given compact connected metric space, X (usually an interval or a finite graph), and the bonding maps are a single function f . In this case the shift maps, σ_N , are iterates of the same automorphism on $\varprojlim \{X, f\}$, and usually is denoted by σ (or σ^N if appropriate).

We are interested in understanding the structure of attractors (under the shift maps) that can exist in inverse limit spaces that arise from implicitly defined difference equations in economics. This is because the forward admissible sequences which make up an attractor under the shift maps are the sequences that we can expect to be ‘seen’. Hence they are the ones the implicitly defined system predict. Unfortunately, there is not a completely accepted definition of attractor, particularly not in the present case under study. The dynamical system $(\varprojlim \{X_{i+n}, f_{i+n}\}, \sigma_n)$ is not “nice” in any conventional sense (it is almost never a smooth map on a manifold). However we will take the following notions of attractor for this paper. These definitions are due to Milnor in [19].

Definition 2.4. Let $f : K \rightarrow K$ be a continuous map of a metric space K . Let $x \in K$. Then the ω -limit set of x is defined to be $\omega_f(x) = \bigcap_{i \in \mathbb{Z}^+} \overline{\{f^m(x) : m \geq i\}}$. Let $A \subseteq K$ be closed and forward invariant, i.e. $f[A] = A$, then the *basin of attraction* of A is defined to be $B(A) = \{x \in K : \omega_f(x) \subseteq A\}$.

We call A a (metric/topological) attractor provided $B(A)$ is *large* (in the sense of measure/topology).

In order to give a precise definition for the basin of attraction of a set A to be topologically large (in the sense of the Baire Theorem) we need a few standard definitions from topology, cf. [16].

Definition 2.5. A subset M of a topological space X is *nowhere dense (NWD)* in X provided the interior of the closure of M in X is empty. We say that M is *meager* in X if M is the countable union of NWD sets in X . If the complement of M is meager in X then we say that M is *residual* in X .

Definition 2.6. Let $f : K \rightarrow K$ be a continuous map of a metric space K . Let $A \subseteq K$ be a closed forward invariant set. Then A is called a *topological attractor* provided $B(A)$ contains a residual subset of an open subset of K and there is no closed forward invariant subset, A' , of A for which $B(A)$ and $B(A')$ coincide up to a meager set.

Definition 2.7. Let $f : K \rightarrow K$ be a continuous map of a metric space K . Let $A \subseteq K$ be a closed forward invariant set. Then A is called a *metric attractor* provided $B(A)$ has positive measure and there is no closed forward invariant subset, A' , of A for which $B(A)$ and $B(A')$ coincide up to a set of measure zero.

An attractor A is called *wild* provided it is a metric attractor but not a topological attractor, cf. [6,9].

Counter to the definition of attractor is that of repeller. We give a definition here that is more general than the standard one for invertible maps.

Definition 2.8. Let $f : M \rightarrow M$ be a continuous map of a metric space M . Let $A \subseteq M$ be closed and forward invariant. Define the *unstable set* of A by $W^u(A) = \{x \in M : \text{there is a sequence } \{x_i\}_{i \in \mathbb{Z}^+} \text{ with } x_0 = x, \text{ such that } f(x_i) = x_{i-1} \text{ for all } i > 0 \text{ and } d(x_i, A) \rightarrow 0 \text{ as } i \rightarrow \infty\}$. We call A a *topological (metric) repelling set* if $W^u(A)$ contains a residual subset of an open set (set of positive measure) and we say that A is a *repeller* provided there is no closed forward invariant subset, A' , of A such that $W^u(A')$ coincides with $W^u(A)$ up to a meager set (set of measure zero).

Definition 2.9. An attractor A is *Liapunov stable* provided there are arbitrarily small neighborhoods, U , of A such that $f(U) \subseteq U$. If A is Liapunov stable and its basin of attraction is open nonempty, then A is *asymptotically stable*.

A necessary ingredient for Definition 2.7 is a measure on $\varprojlim \{X_i, f_i\}$. Unfortunately we have no handy definition of a measure on these spaces. So we focus instead on a quite general class of inverse limit spaces and we examine the question of existence of topological attractors. At the end of the paper we consider a simple case in which a natural measure is present.

Let $\{X_i, f_i\}$ be an inverse sequence of compact metric spaces with the property that $X_i \supseteq X_{i+1}$, there is some map $g : \bigcup_{i \in \mathbb{Z}^+} X_i \rightarrow \bigcup_{i \in \mathbb{Z}^+} X_i$ with finitely many critical points such that $f_i = g|_{X_{i+1}}$ and $\bigcap_{i \in \mathbb{Z}^+} X_i \neq \emptyset$. Notice that this case is slightly more general than that which is often considered in papers on inverse limit spaces, namely inverse sequences with a single connected factor space and a single bonding map. However this is certainly not the most general setting.

A natural candidate for an attractor of the map σ is the set

$$\hat{A} = \varprojlim \{A, g|_A\}$$

where $A \subseteq \bigcap_{i \in \mathbb{Z}^+} X_i$ is a closed and forward invariant *repelling* set. (Notice that since $A \subseteq \bigcap_{i \in \mathbb{Z}^+} X_i$, $\hat{A} \subseteq \varprojlim \{X_{i+n}, f_{i+n}\}$ for each positive integer n .) We mention a few results in this setting.

Lemma 2.10. \hat{A} is closed and $\sigma_n[\hat{A}] = \hat{A}$.

Proof. The fact that \hat{A} is closed follows immediately from the definition of \hat{A} . Let $\hat{x} = (x_1, x_2, \dots) \in \hat{A}$. Choose $n \in \mathbb{Z}^+$. Then $\sigma_n[\hat{x}] = (x_n, x_{n+1}, \dots)$ and by definition $x_{n+i} \in A$ for all i . Hence $\sigma_n[\hat{A}] \subseteq \hat{A}$. Let $\hat{z} = (z_1, z_2, \dots) \in \hat{A}$. Then since $g[A] = A$ and $z_1 \in A$, there is a point $y_1 \in A$ such that $g^n[y_1] = z_1$. Define y_j appropriately for $1 < j < n$ and let $y_n = z_1$ and $y_{n+i+1} = z_i$. Then clearly $\hat{y} = (y_1, y_2, \dots) \in \hat{A}$ and $\sigma_n[\hat{y}] = \hat{z}$. \square

One of the hallmarks of chaotic behavior in a map is the following:

Definition 2.11. Let $F : X \rightarrow X$ be a continuous map of a topological space. We say that F is *topologically transitive* provided that whenever U and V are open sets in X then there is an integer n such that $F^n(U) \cap V \neq \emptyset$.

Assume now that g is topologically transitive on $\bigcap_{i \in \mathbb{Z}^+} X_i$. Let $\widehat{K} = \varprojlim \{\bigcap_{i \in \mathbb{Z}^+} X_i, g\}$. We will show that \hat{A} is not a topological attractor for σ if $A \subsetneq \bigcap_{i \in \mathbb{Z}^+} X_i$ (which is the same as assuming that $\hat{A} \subsetneq \widehat{K}$).

Lemma 2.12. Suppose that g is topologically transitive on $\bigcap_{i \in \mathbb{Z}^+} X_i$. Then $\sigma|_{\widehat{K}}$ is also topologically transitive.

Proof. Let U, V be subsets of \widehat{K} and choose $x, y \in \bigcap_{i \in \mathbb{Z}^+} X_i$, $\varepsilon > 0$ and $M \in \mathbb{N}$ such that $\pi_M^{-1}[B_\varepsilon(x)] \subseteq U$ and $\pi_M^{-1}[B_\varepsilon(y)] \subseteq V$. Choose $n \in \mathbb{N}$ such that $g^n[B_\varepsilon(x)] \cap B_\varepsilon(y) \neq \emptyset$. Then $(\sigma^{-1})^n(U) \cap V \neq \emptyset$. Hence $\sigma^{-1}|_{\widehat{K}}$ is topologically transitive and since $\sigma|_{\widehat{K}} : \widehat{K} \rightarrow \widehat{K}$ is a homeomorphism, $\sigma|_{\widehat{K}}$ is also topologically transitive.

Theorem 2.13. Suppose that g is topologically transitive on $\bigcap_{i \in \mathbb{Z}^+} X_i$. Let $A \subseteq \bigcap_{i \in \mathbb{Z}^+} X_i$ be closed and forward invariant. If $A \neq \bigcap_{i \in \mathbb{Z}^+} X_i$ then \hat{A} is not a topological attractor for σ .

Proof. Let $B = \{U_i\}_{i \in \mathbb{N}}$ be a countable basis for \widehat{K} . Then $\bigcup_{n \in \mathbb{N}} \sigma^{-n}(U_i)$ is open in \widehat{K} . Since σ is topologically transitive on \widehat{K} , $\bigcup_{n \in \mathbb{N}} \sigma^{-n}(U_i)$ is dense in \widehat{K} . So, by the Baire Category Theorem, $\bigcap_{i \in \mathbb{N}} (\bigcup_{n \in \mathbb{N}} \sigma^{-n}(U_i))$ is dense in \widehat{K} . This is obviously the set of points $\hat{x} \in \widehat{K}$ with a dense orbit in \widehat{K} . So the complement of this set is a countable union of nowhere dense sets (i.e. meager) and $B(\hat{A})$ is a subset of the complement of this set. Hence \hat{A} is not a topological attractor for σ . \square

We have the following immediate corollary:

Corollary 2.14. Let $f : [0, 1] \rightarrow [0, 1]$ be topologically transitive. Then there is no closed invariant proper subset, \hat{A} , of $\varprojlim\{[0, 1], f\}$ such that \hat{A} is a topological attractor.

This demonstrates that the spaces which are typically studied by topologists interested in the theory of inverse limit spaces, i.e. inverse limits of topologically transitive interval maps, do not contain smaller subsets which can be topological attractors.

3. Inverse limit spaces of unimodal maps

In this section we move from a discussion of very general inverse limit spaces and turn to an examination of a less general category of functions, specifically unimodal interval maps. The topology of these spaces has been the subject of much study: [2–4, 7, 8, 15, 20–22].

Definition 3.1. Let $F : [a_1, b_1] \rightarrow [a_2, b_2]$ be continuous. Then we say that F is (strictly) *unimodal* provided there is a value $c \in (a_1, b_1)$ such that $F|_{[a_1, c]}$ is (strictly) increasing and $F|_{[c, b_1]}$ is (strictly) decreasing. We call c the *turning point* for F .

The following is an easy lemma, so we have omitted its proof.

Lemma 3.2. Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal. Let $\hat{A} \subseteq \varprojlim\{[0, 1], f\}$ be closed then \hat{A} is σ -invariant if and only if $\hat{A} = \varprojlim\{A, f|_A\}$ where $A = \pi_1(\hat{A})$ and A is f -invariant.

Theorem 3.3. Let $f : [0, 1] \rightarrow [0, 1]$ be unimodal. Let \hat{A} be closed and σ -invariant with $\pi_1(\hat{A}) = A$. Suppose that $f^{-1}(A) \neq A$. Then \hat{A} is not Liapunov stable, and therefore \hat{A} is not asymptotically stable.

Proof. Let $x \in A$ with $f^{-1}(x) \setminus A \neq \emptyset$. Let $x' \in f^{-1}(x) \setminus A$. Since A is closed and $x' \notin A$ let $\gamma > 0$ be defined so that $B_\gamma(x') \cap A = \emptyset$. Let $\varepsilon > 0$ with $\varepsilon < \frac{\gamma}{2}$ and let U be an ε -neighborhood of \hat{A} . Let $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$. Let $\hat{z} = (z_0, z_1, \dots, z_n, \dots) \in \hat{A}$ be defined so that $z_n = x$ and $z_i = f^{n-i}(x)$ for $0 \leq i \leq n$ and z_i defined appropriately

for $i > n$. Let $\hat{z}' = (z_0, z_1, \dots, z_n, x', z'_{n+2}, \dots)$ with the z'_{n+i} the appropriate preimages of x' . Then $\hat{z}' \notin \hat{A}$, but $d(\hat{z}, \hat{z}') \leq \frac{1}{2^n} < \varepsilon$ so $\hat{z}' \in U$. Notice however though that $\sigma^n(\hat{z}')$ has first coordinate x' and $|x' - w| > \gamma$ for all $w \in A$. So if $\hat{w} = (w_1, w_2, \dots) \in \hat{A}$ then $|w_1 - x'| > \gamma$ so $d(\hat{z}', \hat{w}) > \frac{\gamma}{2} > \varepsilon$. Thus $\sigma^n(U) \not\subseteq U$ and \hat{A} is not Liapunov stable. \square

It immediately follows that if the f -invariant set, A , is a periodic orbit of period $n \geq 3$ then $\hat{A} = \varprojlim \{A, f|_A\}$ cannot be a Liapunov stable attractor (and hence it cannot be an asymptotically stable attractor.)

There are essentially three ‘types’ of maps that we will consider:

Definition 3.4. Let f be a map with domain $[0, 1]$. Then we say that f is

- (1) a *type A* unimodal map provided f is strictly unimodal on $[0, 1]$ with turning point c , $f(0) = 0$ and $f(c) \leq 1$,
- (2) a *type B* unimodal map provided f is strictly unimodal on $[0, 1]$ with turning point c , $f(0) > 0$ and $f(c) \leq 1$,
- (3) a *type C* map provided there is a point $c \in (0, 1)$ such that $f|_{[0,c]}$ is strictly increasing and $f|_{[c,1]}$ is strictly decreasing, $f(0) = f(1) = 0$, and $f(c) > 1$.

One case of unimodal map that we do not consider in detail is one in which $f(c) < c$, where c is the unique turning point. In this case the non-zero fixed point p is in the interval $[0, c]$ and $\varprojlim \{[0, 1], f\}$ is homeomorphic to $\varprojlim \{[0, p], f|_{[0,p]}\}$. Since f is monotone on $[0, p]$ we have that the inverse limit space is simply an arc, and so the action of σ on $\varprojlim \{[0, 1], f\}$ is simply the action of a homeomorphism of an arc.

Notice that if f is a type A unimodal map then f is surjective onto the interval $I = [0, f(c)]$ and if $z \in (f(c), 1]$ then $f^{-1}(z) = \emptyset$. Hence $\varprojlim \{[0, 1], f\}$ is homeomorphic to $\varprojlim \{I, f|_I\}$. Also if f is a type B unimodal map then f is surjective onto the interval $J = [f^2(c), f(c)]$, and again $\varprojlim \{[0, 1], f\}$ is homeomorphic to $\varprojlim \{J, f|_J\}$. So without loss of generality we will assume that if f is either a type A or a type B unimodal map then $f(c) = 1$ and $I = [0, 1]$ or $J = [0, 1]$.

Maps such as those represented in Figs. 2 and 3 arise in the context of the economic problem mentioned in Section 1 for a variety of reasonable utility functions.

We state and prove two theorems regarding the existence of single points in the inverse limit space which are also asymptotically stable attractors, first in the type A case and then in the type B case.

Theorem 3.5. Let f be a type A map, $[0, 1]$, and $\hat{0} = (0, 0, \dots)$. If $f^2(c) = f(1) > 0$ and $f'(0) > 1$ then $\hat{0}$ is an asymptotically stable attractor. Moreover if $f(x) > x$ for all $x \in (0, c)$ then $\hat{0}$ is the only topological attractor for σ on $\varprojlim \{[0, 1], f\}$.

Proof. Let $x \in [0, 1]$. Since f is increasing on $[0, c]$ and since it maps the interval $[0, c]$ onto $[0, 1]$, x has a unique preimage in $[0, c]$ and it is closer to 0 than x is. In fact the sequence of these preimages will converge to 0. Let $\varepsilon > 0$ such that $\varepsilon < f^2(c)$. Let $U = \pi_1^{-1}([0, \varepsilon))$. Then U is an open subset of $\varprojlim \{[0, 1], f\}$. Let $\hat{x} = (x_1, x_2, \dots) \in U$. Then

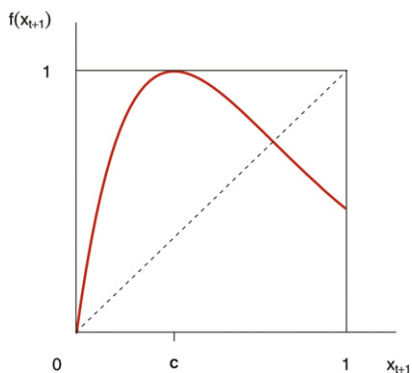


Fig. 2. A type A map.

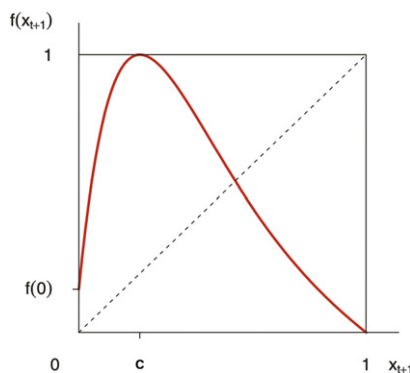


Fig. 3. A type B map.

$x_1 \in [0, \varepsilon)$ and there is no $z \in [c, f(c)]$ with $f(z) = x_1$. Hence $x_2 \in [0, c)$ and in fact x_2 is closer to 0 than x_1 is. So $x_2 \in [0, \varepsilon)$. In fact it is easy to see that $x_n \in [0, \varepsilon)$ for all $n \in \mathbb{N}$ and $x_n \rightarrow 0$ as $n \rightarrow \infty$. Thus $\sigma^n(\hat{x}) \rightarrow \hat{0}$ as $n \rightarrow \infty$. Thus $U \subseteq B(\hat{0})$ and $\hat{0}$ is an attractor for $\varprojlim\{[0, 1], f\}$. It is also easy to see that $\sigma(U) \subseteq U$ and $\bigcap_{n \in \mathbb{N}} \sigma^n(U) = \{\hat{0}\}$ so $\hat{0}$ is Liapunov stable.

To see that $B(\hat{0})$ is open notice that if $\hat{z} \in B(\hat{0})$ then there is an integer n such that $z_n \in [0, \varepsilon)$. Let $U_n = \pi_n^{-1}([0, \varepsilon))$. Then U_n is an open set in $\varprojlim\{[0, 1], f\}$ and $B(\hat{0}) = \bigcup_{i \in \mathbb{Z}^+} U_i$ and therefore $\hat{0}$ is asymptotically stable.

Assume now that $f(x) > x$ for all $x \in (0, c)$. To show that there is no other attractor for $\varprojlim\{[0, 1], f\}$ we will demonstrate that every open set in $\varprojlim\{[0, 1], f\}$ meets $B(\hat{0})$. Let $\hat{z} = (z_1, z_2, \dots) \in \varprojlim\{[0, 1], f\} \setminus B(\hat{0})$ and $\gamma > 0$. Let $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \gamma$. Since $\hat{z} \notin B(\hat{0})$ there is some integer $M > n$ such that z_M is not the preimage of z_{M-1} in the interval $[0, c)$ (because if every coordinate was always this preimage then $\sigma^n(\hat{z}) \rightarrow \hat{0}$). Let $z'_M = z_M$ and for each $i \in \mathbb{N}$ let z'_{M+i} be the preimage of $z'_{M+(i-1)}$ in $[0, c)$. Let $\hat{z}' = (z_1, z_2, \dots, z_n, \dots, z'_M, z'_{M+1}, \dots, z'_{M+i}, \dots) \in \varprojlim\{[0, 1], f\}$. Notice that $d(\hat{z}, \hat{z}') \leq \frac{1}{2^M} < \gamma$ but $\sigma^n(\hat{z}') \rightarrow \hat{0}$. So $B(\hat{0})$ is open and dense in $\varprojlim\{[0, 1], f\}$ and thus $\varprojlim\{[0, 1], f\} \setminus B(\hat{0})$ is closed NWD. \square

If instead we had $f^2(c) = f(1) = 0$ then the standard models (such as the quadratic maps or the tent maps) are topologically transitive and so $\varprojlim\{[0, 1], f\}$ contains no proper subsets which are topological attractors for σ by Corollary 2.14; instead the only asymptotically stable attractor (for both σ and σ^{-1}) is (trivially) the entire space $\varprojlim\{[0, 1], f\}$. Thus for the type A maps (at least the typical type A maps) we have a good understanding of the structure of topological attractors. The situation is not so simple in the case of type B maps.

Theorem 3.6. *Let $f : [0, 1] \rightarrow [0, 1]$ be a type B unimodal map with unique fixed point $p \in [c, 1]$ that is repelling on $[c, 1]$ and suppose that $f(0) > p$. Then the point $\hat{p} = (p, p, \dots) \in \varprojlim\{[0, 1], f\}$ is an asymptotically stable attractor, and it is the only topological attractor in $\varprojlim\{[0, 1], f\}$.*

Proof. Let $\varepsilon > 0$ be defined so that $f(0) > p + \varepsilon$. Let $U = \pi_1^{-1}((p - \varepsilon, p + \varepsilon))$. Then U is an open set in $\varprojlim\{[0, 1], f\}$ and $\hat{p} \in U$. Let $\hat{z} = (z_1, z_2, \dots) \in U$. Then $z_1 \in (p - \varepsilon, p + \varepsilon)$ and since $f(0) > z_1$ the only preimage of z_1 under f is $z_2 \in (p - \varepsilon, p + \varepsilon)$. Since p is repelling on $[x^*, 1]$ it must be the case that z_2 is closer to p than z_1 is. Hence $\sigma(U) \subseteq U$ and \hat{p} is Liapunov stable. Moreover the sequence $z_i \rightarrow p$ as $i \rightarrow \infty$ since p is repelling. Thus for any $\hat{z} \in U$, $\sigma^n(\hat{z}) \rightarrow \hat{p}$ as $n \rightarrow \infty$. Thus $U \subseteq B(\hat{p})$ and \hat{p} is an attractor.

As before we see that \hat{p} is asymptotically stable by noticing that if $\hat{z} \in B(\hat{p})$ then there is an integer n so that $z_n \in (p - \varepsilon, p + \varepsilon)$. Let $U_n = \pi_n^{-1}((p - \varepsilon, p + \varepsilon))$. It is easy to see then that $B(\hat{p}) = \bigcup_{i \in \mathbb{Z}^+} U_i$ which is an open set.

We will show that \hat{p} is the only attractor by showing that $B(\hat{p})$ is dense in $\varprojlim\{[0, 1], f\}$. The result will follow because no other set K could possibly contain a nonempty open set in $B(K)$ because in every nonempty open set there will be points of $B(\hat{p})$. Let $\hat{x} = (x_1, x_2, \dots) \in \varprojlim\{[0, 1], f\}$ and let $\gamma > 0$. Let $n \in \mathbb{N}$ so that $\frac{1}{2^n} < \gamma$ and define x'_{n+1} to be the preimage of x_n in the interval $[x^*, 1]$ (recall that since f is type B the interval $[x^*, 1]$ is mapped onto $[0, 1]$ so each point has such a preimage). Given x'_{n+i} let x'_{n+i+1} be the unique preimage of x'_{n+i} in the interval $[x^*, 1]$. Notice that since p is repelling on $[x^*, 1]$ we have $x'_{n+i} \rightarrow p$ as $i \rightarrow \infty$. Let $\hat{x}' = (x_1, x_2, \dots, x_n, x'_{n+1}, \dots, x'_{n+i}, \dots) \in \varprojlim\{[0, 1], f\}$. Then clearly $d(\hat{x}, \hat{x}') < \gamma$ and $\sigma^n(\hat{x}') \rightarrow \hat{p}$. Thus $B(\hat{p})$ is a dense open set in $\varprojlim\{[0, 1], f\}$. \square

Notice, of course, that it is possible to have a type B map with $f(0) > p$ and p not repelling on $[c, 1]$ or even to have a period two orbit contained in $[c, 1]$ that is repelling which could generate a two-point set in $\varprojlim\{[0, 1], f\}$ that is an asymptotically stable attractor. If though the map is some “modified” logistic map such as $f(x) = k + \mu x(1 - x)$ with critical point c with $f^2(c) < c < f(c)$ then we can show that $|f'(p)| > 1$. So in this canonical case the fixed point will be repelling.

Next we consider $f : [0, 1] \rightarrow [0, 1]$ a type B unimodal map with $f(0) \leq p$. This is a staggeringly rich collection of maps which generate many different types of inverse limit spaces. All of them contain an indecomposable subcontinuum and if $f(0) < p$ then the inverse limit is itself an indecomposable continuum, [12]. For the sake of brevity we analyze attractors in inverse limits of fairly simple type B unimodal maps with $f(0) < p$. Specifically we assume that f has a stable attractor. This occurs for many unimodal maps with negative Schwarzian derivative, cf. [14].

Let $f: [0, 1] \rightarrow [0, 1]$ be a type B map with a stable topological attractor, P , then by [14] there are n open intervals A_0, A_1, \dots, A_{n-1} each with $f^i(A_0) \subseteq A_i$ such that $\bigcup_{i \in \mathbb{N}} A_i$ is the stable manifold of P . Then we have that $f[\bigcup_{i=0}^{n-1} A_i] \subseteq \bigcup_{i=1}^{n-1} A_i$. Let $\Lambda = \{x \in [0, 1]: f^n(x) \notin \bigcup_{i=1}^n A_i\}$. Then $f|_\Lambda$ is conjugate to a subshift of finite type σ_A on some sequence space Σ_A , and there is a decomposition of Λ into a disjoint union of Cantor sets, Λ_i , such that $f|_{\Lambda_i}$ is topologically transitive. For simplicity, in this paper we assume that f restricted to Λ is topologically transitive, and in a forthcoming paper we consider the more general case. Let $\hat{\Lambda} = \varprojlim \{\Lambda, f|_\Lambda\}$.

Lemma 3.7. $f^{-1}(\Lambda) = \Lambda$.

Lemma 3.8. $\hat{\Lambda}$ is a Cantor set in $\varprojlim \{[0, 1], f\}$ and $\sigma(\hat{\Lambda}) = \sigma^{-1}(\hat{\Lambda}) = \hat{\Lambda}$.

Proof. Notice that $\hat{\Lambda} \subseteq \prod_{i=1}^\infty \Lambda$ so it is totally disconnected and since Λ is compact $\hat{\Lambda}$ is compact. To see that $\hat{\Lambda}$ is perfect let $\hat{x} = (x_1, x_2, \dots) \in \hat{\Lambda}$ and let $\varepsilon > 0$. By the uniform continuity of f there is an integer N and a positive number δ such that if $\hat{z} = (z_1, z_2, \dots) \in \varprojlim \{[0, 1], f\}$ has $d(z_N, x_N) < \delta$ then $\hat{d}[\hat{z}, \hat{x}] < \varepsilon$. Since there is clearly such a $\hat{z} \in \hat{\Lambda}$ we have that $\hat{\Lambda}$ is perfect and $\hat{\Lambda}$ is a Cantor set.

We show that $\sigma(\hat{\Lambda}) \subseteq \hat{\Lambda}$ and that $\sigma^{-1}(\hat{\Lambda}) \subseteq \hat{\Lambda}$. The result will follow. Let $\hat{x} = (x_1, x_2, \dots) \in \hat{\Lambda}$. Then clearly both $\sigma(\hat{x}) = (x_2, x_3, \dots)$ and $\sigma^{-1}(\hat{x}) = (f(x_1), x_1, \dots) \in \hat{\Lambda}$. \square

Lemma 3.9. $\hat{z} = (z_1, z_2, \dots) \notin \hat{\Lambda}$ if, and only if there is an $n \in \mathbb{N} \cup \{0\}$ such that $f^n(z_1) \in \bigcup_{i=0}^{n-1} A_i$.

Proof. If $\hat{z} \notin \hat{\Lambda}$ then there is an integer m so that $z_m \notin \Lambda$ so there is an integer n_1 so that $f^{n_1}(z_m) \in \bigcup_{i=0}^{n_1-1} A_i$. If $n_1 \leq m$ then $z_1 \in \bigcup_{i=1}^{n_1-1} A_i$ and if $n_1 > m$ then $f^{n_1-m}(z_1) \in \bigcup A_i$. The other direction is easier. \square

Lemma 3.10. If $\hat{z} \notin B(\hat{\Lambda})$ then for every $n \in \mathbb{N}$, $z_n \in \bigcup A_i$.

Proof. Let $\hat{z} \notin B(\hat{\Lambda})$. We will show that for every integer n there is an integer $m > n$ so that $z_m \in \bigcup A_i$. The result will follow.

Let $n \in \mathbb{N}$ and $\hat{x} \in \omega(\hat{z}) \setminus \hat{\Lambda}$. Let N be the integer guaranteed by the previous proposition so that $f^N(x_1) \in \bigcup A_i$. Let $\varepsilon_0 > 0$ be defined so that $B_{\varepsilon_0}[f^N(x_1)] \subseteq \bigcup A_i$ and let $\delta > 0$ so that if $|w - x_1| < \delta$ then $|f^N(w) - f^N(x_1)| < \varepsilon$. Let $\varepsilon_1 < \frac{\delta}{2}$ and choose $m \geq n + N$ so that $\hat{d}[\sigma^m(\hat{z}), \hat{x}] < \varepsilon_1$. Then clearly $\frac{|z_m - x_1|}{2} < \varepsilon_1$ so $|z_m - x_1| < \delta$. Hence $z_{m-N} = f^N(z_m) \in B_{\varepsilon_0}(f^N(x_1)) \subseteq \bigcup A_i$ and by our choice of N and m , $m - N \geq n$. \square

Let $A = \bigcup_{i=0}^{n-1} \overline{A_i} \supseteq \bigcup A_i$ and let $\hat{A} = \varprojlim \{A, f|_A\}$. Then we have the following:

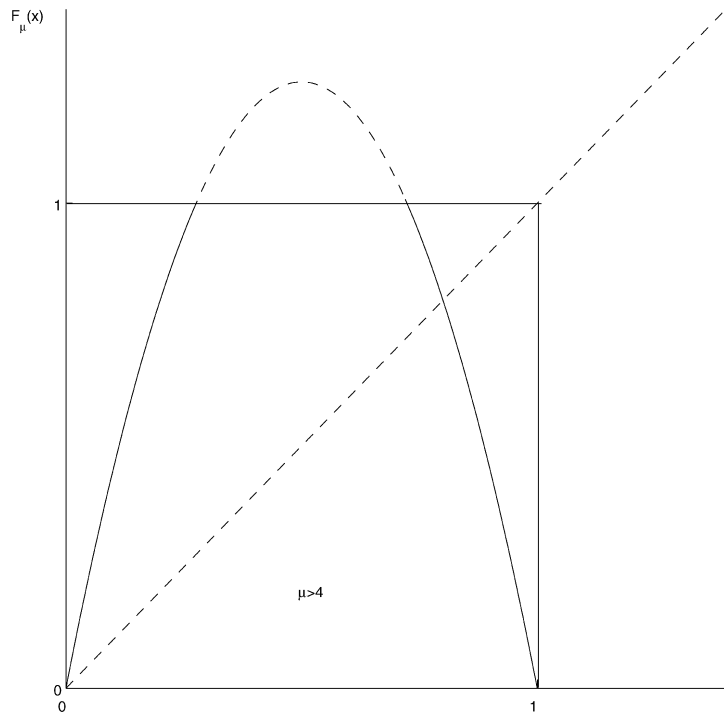
Lemma 3.11. $B(\hat{\Lambda}) \supseteq (\varprojlim \{[0, 1], f\} \setminus \hat{\Lambda})$.

Proof. By the previous proposition if $\hat{z} \notin B(\hat{\Lambda})$ then $z_i \in \bigcup A_i \subseteq A$ for all $i \in \mathbb{N}$. So if $\hat{z} \notin B(\hat{\Lambda})$ then $\hat{z} \in \hat{A}$. Thus every point of $\varprojlim \{[0, 1], f\} \setminus \hat{\Lambda}$ is in $B(\hat{\Lambda})$. \square

Theorem 3.12. If $f|_\Lambda$ is topologically transitive, then $\hat{\Lambda}$ is a topological attractor for σ on $\varprojlim \{[0, 1], f\}$. Moreover, $\hat{\Lambda}$ is the only topological attractor.

Proof. Clearly $\hat{A} = \bigcup \hat{A}_i$ where \hat{A}_i is the inverse limit defined by $\varprojlim \{\overline{A_i}, f^n|_{\overline{A_i}}\}$ which is a proper subcontinuum of $\varprojlim \{[0, 1], f\}$. Since $\varprojlim \{[0, 1], f\}$ is indecomposable each \hat{A}_i is nowhere dense and so \hat{A} is meager. Thus $B(\hat{\Lambda})$ contains a residual subset of $\varprojlim \{[0, 1], f\}$, an open set. Notice also that since $f|_\Lambda$ is topologically transitive, $\sigma|_{\hat{\Lambda}}$ is topologically transitive.

Let \hat{K} be another closed σ -invariant subset of $\varprojlim \{[0, 1], f\}$. Then $\pi_1[\hat{K}] = K$ is closed and f -invariant. So $K \subseteq A \cup \Lambda$, [14]. In either case, if $K \neq \Lambda$, $B(\hat{K})$ is NWD. \square

Fig. 4. The logistic map F_μ .

We end the paper with the familiar logistic map $F_\mu(x) = \mu x(1 - x)$ with $\mu > 4$ as our ‘canonical model’ of a type C map. (See Fig. 4.)

Let $X_1 = [0, 1]$, $X_2 = X_1 \cap F_\mu^{-1}[X_1]$ and inductively define $X_i = X_{i-1} \cap F_\mu^{-1}[X_{i-1}]$. Also define $f_i : X_{i+1} \rightarrow X_i$ by $f_i = F_\mu|_{X_{i+1}}$. Then $\{X_i, f_i\}$ is an inverse sequence and $\varprojlim \{X_i, f_i\}$ is the space consisting of all possible forward admissible sequences, (x_1, x_2, \dots) , permitted by the difference equation with $x_i \in [0, 1]$.

Let 2^ω denote the Cantor set $\{(z_0, z_1, \dots) : z_i \in \{0, 1\} \text{ for each } i \in \mathbb{N}\}$. Consider the standard itinerary mapping $i(x) = 0$ if, and only if $x < 1/2$ and $i(x) = 1$ otherwise. Let Λ be the forward F_μ -invariant Cantor set in $[0, 1]$, i.e. $\Lambda = \bigcap_{n \geq 0} F_\mu^{-n}([0, 1])$. Let $\hat{\Lambda} = \varprojlim \{\Lambda, F_\mu\}$. Notice that $\hat{\Lambda}$ is a homeomorphic embedding of Λ in $\varprojlim \{X_i, f_i\}$. Recall from Lemma 2.10 that $\hat{\Lambda}$ is a closed and σ invariant subset of $\varprojlim \{X_i, f_i\}$.

We begin by showing that $\varprojlim \{X_i, f_i\}$ is homeomorphic to the product of a Cantor set and an arc.

Lemma 3.13. Define $h : \varprojlim \{X_i, f_i\} \rightarrow [0, 1] \times 2^\omega$ by

$$h[(x_0, x_1, \dots)] = (x_0, (i[x_1], i[x_2], \dots)).$$

Then h is a homeomorphism.

Proof. Let $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots) \in \varprojlim \{X_i, f_i\}$ such that $h(x) = h(y)$. Then $x_0 = y_0$ and since each point has exactly two preimages, one less than $1/2$ and one greater than $1/2$ and since $i[x_1] = i[y_1]$ it must be the case that $x_1 = y_1$. Continuing we see that in fact $x_n = y_n$ for all $n \in \mathbb{N}$. Hence $x = y$ and h is injective. Let $z \in [0, 1] \times 2^\omega$. Choose $w \in \varprojlim \{X_i, f_i\}$ such that $w_0 = z_0$ and $i[w_i] = z_i$ for each $i > 0$. Then clearly $h(w) = z$ and h is surjective. Notice that $\pi_0 \circ h$ is simply the identity so it is continuous and for each integer $j > 0$, $\pi_j \circ h = i \circ \pi_j$ it follows easily that $i \circ \pi_j$ is continuous so we have shown that for each integer k , $\pi_k \circ h$ is continuous. It follows that h is continuous and hence a homeomorphism. \square

Lemma 3.14. Let $z \in \varprojlim \{X_i, f_i\}$ then $\sigma^n(z) \rightarrow \hat{\Lambda}$ as $n \rightarrow \infty$.

Proof. To see this notice that $\bigcap_{n \geq 0} X_n = \Lambda$ and recall that $\pi_j \circ \sigma = \pi_{j+1}$ so $\pi_0 \circ \sigma^n = \pi_n$. So for each $\varepsilon > 0$ there is an integer n such that $X_n \subseteq B_H(\Lambda, \varepsilon)$, where by $B_H(\Lambda, \varepsilon)$ we mean the ε -disc about Λ in the Hausdorff metric. Since $\pi_0 \circ \sigma^n = \pi_n$ we can choose N large enough to make the 0th coordinate of $\sigma^N(z)$ as close to Λ as we like (in the Hausdorff metric). It follows that $\sigma^n(z) \rightarrow \hat{\Lambda}$ as $n \rightarrow \infty$. \square

Now consider \hat{K} a closed invariant proper subset of $\hat{\Lambda}$. Since $F_\mu|_\Lambda$ is topologically transitive, we have from Theorem 2.13 that the basin of attraction of \hat{K} , $B(\hat{K})$, is NWD. Hence \hat{K} is not a topological attractor and $\hat{\Lambda}$ is a topological attractor. In fact from Lemma 3.14 we can derive an even stronger proposition, namely:

Lemma 3.15. *$\hat{\Lambda}$ is an asymptotically stable attractor.*

Proof. By the previous lemma we see that $B(\hat{\Lambda}) = \varprojlim \{X_i, f_i\}$. Hence $B(\hat{\Lambda})$ is open nonempty. \square

If we take ν to be the product of Lebesgue measure, λ , on $[0, 1]$ and the Lebesgue product measure, $\hat{\lambda}$, on $2^\omega = \{0, 1\}^\mathbb{N}$ and we define $\hat{\nu}[\hat{K}] := \nu[h(\hat{K})]$ for all subsets \hat{K} of $\varprojlim \{X_i, f_i\}$, then it is clear that $\hat{\nu}[B(\hat{\Lambda})] = 1$.

Lemma 3.16. *Let $K \subsetneq \Lambda$ be closed and invariant. Then $\hat{\nu}[B(\hat{K})] = 0$.*

Proof. In this proof we identify Λ with the set 2^ω and we lose no generality in making this identification since f_i restricted to Λ is topologically conjugate to the full shift on 2^ω . Notice that since $\hat{\lambda}$ is ergodic on 2^ω and K is a closed invariant proper subset of Λ , $\hat{\lambda}(K) = 0$. Hence $\hat{\nu}(\hat{K}) = 0$. Let $\Delta \subset \Lambda$ be the collection of all points in Λ with dense orbits. It is easy to see that $\hat{\lambda}(\Delta) = 1$ and $\Delta \cap B(K) = \emptyset$. Hence $\hat{\lambda}[B(K)] = 0$ and $\hat{\nu}[B(\hat{K})] = 0$. \square

As a result of these propositions we have the following theorem:

Theorem 3.17. *Let $\mu > 4$. Let $K \subseteq [0, 1]$ such that K is closed invariant and repelling under F_μ . Let $\hat{K} = \varprojlim \{K, F_\mu|_K\}$. Then \hat{K} is an attractor for $\varprojlim \{X_i, f_i\}$ if, and only if $K = \Lambda$.*

Obviously this result generalizes easily to multimodal maps with invariant Cantor sets represented by k^ω for some positive integer k rather than just 2^ω .

Note. Recently we have become aware of a very nice paper by Kennedy, Stockman and Yorke [17]. In this paper the authors also mention the problem of backwards dynamics in economics and the fact that the collection of all solutions to such a problem form an inverse limit space. A key difference in our paper is that we seek to use the dynamics implicit in the inverse limit space to describe the *predicted* solutions which are implicit in these economic models, and we also are considering a broader collection of functions (i.e. the unimodal maps of type B).

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